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Areas of Indifference: Measures to compare similar portfolios

Investments

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Background

An efficient frontier is a visual representation of a portfolio's risk and return characteristics. However, the efficient frontier does not represent the only optimal set of investment opportunities available, given that it is based on the estimated expected return of all underlying assets, their interactions and imposed constraints. Furthermore, it is quite possible that changing the structure of a portfolio to be more "optimal" may result in high turnover costs that are not compensated for.

A framework for solutions to these problems is to consider an *area of indifference* coupled with random portfolio generation techniques. This means that when investigating how to improve a portfolio, we can consider a broad range of portfolios that have similar statistical properties, in a range around the current portfolio that extends towards the efficient frontier. This allows us to identify whether the proposed changes will improve the expected outcomes with sufficient reliability, after the change in costs is accounted for.

Introduction

An area of indifference compares the characteristics of a base portfolio to the characteristics of a set of proposed target portfolios. A numerical value is then used to represent how these portfolios differ to the base portfolio. If the value calculated falls within a given range, then as investors we would feel indifferent towards the portfolios. On an efficient frontier, we can imagine this portfolio as a point moving away from a given reference point (or baseline portfolio) – as the gap widens – so that the statistical similarity between the portfolios decreases until they are sufficiently different to trigger a rebalance action. The number should fall between 0% and 100%, where 0% indicates no difference and 100% indicates that the expected performance of the different portfolio is completely different. A mathematical overview of the area of indifference is given in the Appendix at the end of the document.

Motivation for areas of indifference

Often in portfolio construction, we construct an efficient frontier for clients, given their investment requirements and constraints, and it is noted how close their current portfolio is to this theoretically optimal frontier. The questions we answer here are

- by how much (if at all) is the current portfolio deficient against the optimal portfolio? And
- how can we improve our current portfolio to achieve a more optimal state while taking turnover and trading costs into account?

Additionally, we can answer other questions without the need for an efficient frontier, such as:

- Is it worth including additional asset x into the portfolio?
- How should asset x be included in the portfolio so as not to perturb the portfolio structure too much?
- How should the performance or statistical properties of asset x change in order for it to be included in a portfolio?
- In a multi-period context, what are the different portfolio options that will ensure the portfolio meets specific criteria over the longer term?

Constructing different portfolio options, and thus the area of indifference, requires generating a set of random portfolios that adheres to the investment constraints of the current portfolio. Additional advantages to comparing portfolios using indifference measures with random portfolio generation include:

- Portfolios can be non-normal, as can the underlying components.
- The statistical foundation of indifference measures is robust.

In presenting the results, we characterise the portfolio performance using the mean and standard deviation of returns as a metric. Note, however, that the calculation of our results on the indifference measure is performed using all the distributional characteristics of the portfolio.

Comparing different divergence measures from a reference portfolio

The structure identified below is used as a reference portfolio. The asset distributions are generated from the latest 15-year history of monthly returns in Rand terms from the associated proxies.

Table 1.

Asset Class	Proxy or asset	Current Weight	Max Weight	Historical Return p.a.	Mean	Standard Deviation
Domestic Inflation Linked Bonds	Barclays SA ILB Index	5%	15%	8.71%	8.53%	5.36%
Domestic Bonds	All Bond Index	15%	100%	8.59%	8.52%	7.08%
Cash	STFIND (Cash Composite)	10%	50%	7.36%	7.13%	0.46%
Domestic Listed Equity	J433T (Swix Capped Index)	30%	75%	15.10%	15.09%	13.83%
Listed Property	J253T (SA Listed Property Index)	10%	20%	16.81%	16.92%	16.12%
Foreign Equity	MSCI ACWI	20%	30%	12.40%	12.71%	14.08%
Foreign Bonds	Bloomberg Barclays Global Aggregate Bond Index	5%	20%	8.79%	9.55%	14.95%
Foreign Bonds	S&P Global Property Index	3%	10%	13.09%	13.96%	17.88%
Foreign Infrastructure	MSCI World Core Infrastructure Index	2%	10%	14.96%	14.95%	13.72%

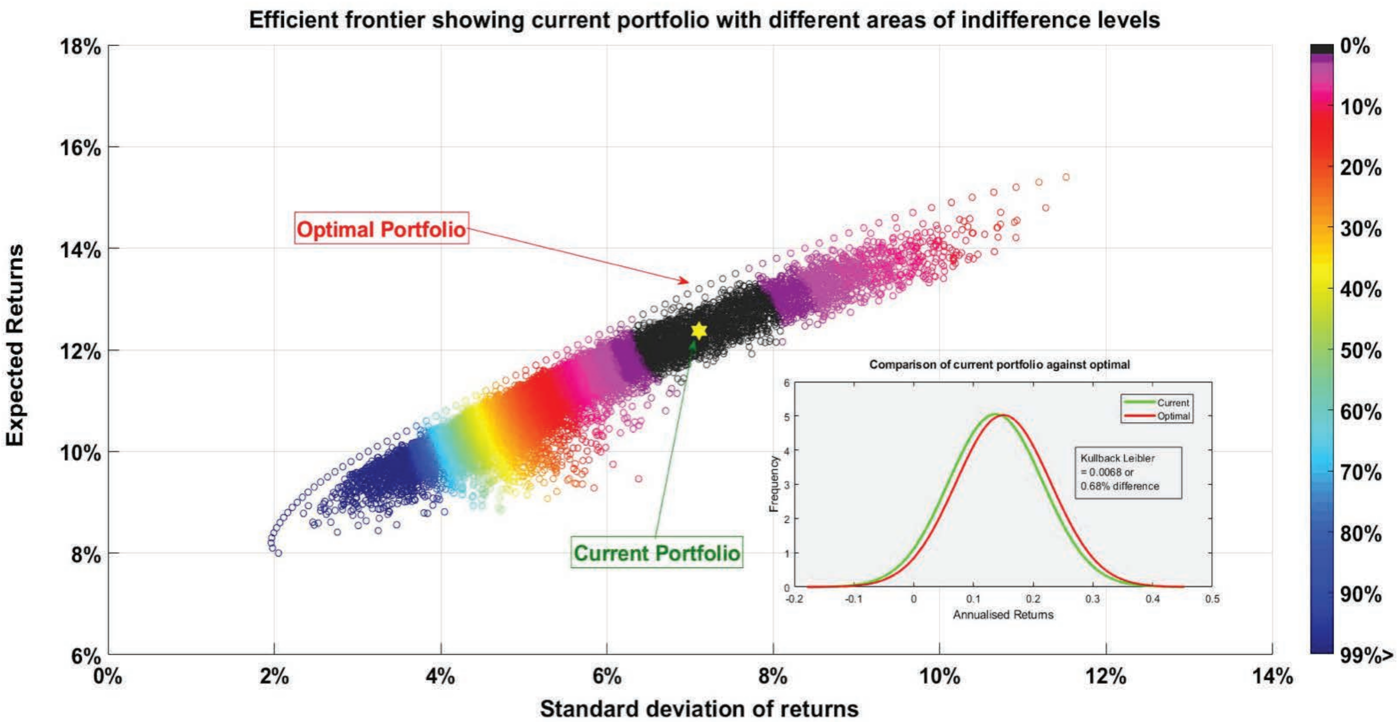
In our **fictitious portfolio**, we will enforce Regulation 28 constraints as follows:

- Total international asset exposure must be less than 30%
- Listed equity (SWIX plus MSCI ACWI) must be less than 75%
- The sum of all properties, local and foreign, may not add up to more than 25%.

From an investability perspective, we also limited the total exposure to foreign bonds, property and infrastructure.

The efficient frontier for the combination of assets, the current portfolio, a generation of 20000 random portfolios satisfying the constraints as well as the Kullback-Leibler divergence measure (described in the Appendix), with our portfolio as the reference portfolio, is shown below:

Figure 1.1



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The figure shows the current portfolio, together with the efficient frontier and set of random portfolios (that satisfy the investment constraints). The various shades of colour show how similar different portfolios are and we refer to the region where all portfolios at 2%+ different as an area of indifference. The area of indifference is plotted relative to the current portfolio.

We also show the most optimal portfolio relative to the current portfolio, for a given level of risk. We also provide a smaller graph showing the statistical spread of returns between the optimal portfolio and the current portfolio and here we can see that the optimal portfolio looks very similar, with a higher mean, to the current portfolio. The two portfolios differ approximately 0.68% and as such would give similar statistical returns.

We will now look more specifically at areas of indifference for the above portfolio and for the optimal efficient frontier portfolio and we will show the area of indifference at a finer level. We show the areas of indifference as concentric circles varying in 0.5% levels. The current portfolios concentric circles are plotted in colour and the optimal portfolio on the efficient frontier is plotted in black stripes:

Figure 1.2

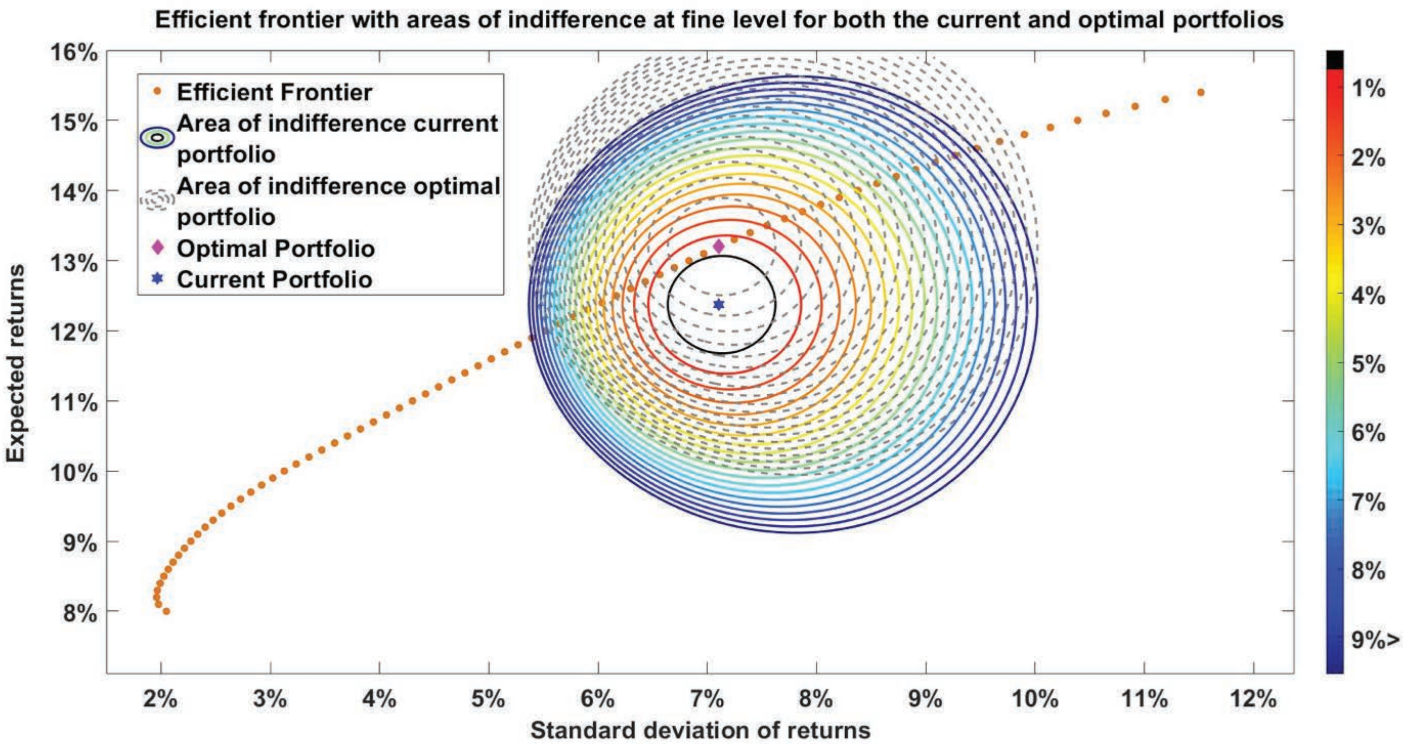

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Figure 1.2 shows that there are various areas of indifference and that within a 0.5% difference range, there are many portfolios that are equivalent to ours. This also hold for the optimal portfolio and we see that there is an overlapping region where a set of portfolios exist that fall within a 0.5% level of indifference of 0.5% of both the current and optimal portfolios. If we would like to change our current portfolio, then we are likely to search for an appropriate portfolio in this region.

Furthermore, the various areas of indifference show that if we were to change the structure of our portfolio to any other portfolio's structure that falls within the specified level of indifference, it would not be worth changing to. Only portfolios that fall outside our required level of indifference will be worth changing to.

We can now start to answer interesting questions such as which portfolio will, within a 0.5% level of indifference, improve our portfolio's return while minimising turnover subject to various investment costs?

Further then to this, what is the probability attached to this optimal improvement? In other words, if we change the portfolio structure to a different structure for an improvement of 100bps, then what is the likelihood of the portfolio actually achieving this additional return?

Considering the costs involved

As an example, we pursued a maximum turnover target of 40% (or one-sided 20%) and show the following illustrative[1] schedule of fees (in bps):

Table 2: Buy-side fees:

Proxy or asset	Bid/offer spread	Brokerage Fees	Securities Transfer Tax	Entry Fee
Domestic Inflation Linked Bonds	10bps	5bps	0bps	0bps
Domestic Bonds	5bps	0.5bps	0bps	0bps
Cash	15bps	0bps	0bps	0bps
Domestic Listed Equity	15bps	10bps	25bps	0bps
Listed Property	15bps	10bps	25 bps	0bps
Foreign Equity	20bps	20bps	30bps	0bps
Foreign Bonds	10 bps	10 bps	0bps	0bps
Foreign Bonds	20bps	20bps	25bps	0bps
Foreign Infrastructure	15bps	0bps	0bps	100bps

Table 3: Sell-side fees

Proxy or asset	Bid/offer spread	Brokerage Fees	Exit Fee
Domestic Inflation Linked Bonds	10bps	5bps	0bps
Domestic Bonds	5bps	0.5bps	0bps
Cash	15bps	0bps	0bps
Domestic Listed Equity	15bps	10bps	25bps
Listed Property	15bps	10bps	25 bps
Foreign Equity	20bps	20bps	30bps
Foreign Bonds	10 bps	10 bps	0bps
Foreign Property	20bps	20bps	0bps
Foreign Infrastructure	15bps	0bps	200bps

[1] The values in the table are illustrative and should not be taken to be absolute. In practice there may be additional fees included and such fees vary on a variety of factors such the liquidity and marketability of the securities in question. The figures are provided for illustrative purposes pertaining to the scope of the example.

From these input cost tables, we now seek a portfolio that maximises the additional improvement in mean returns per turnover cost (within our maximum turnover) that lies between an indifference level of 0.5% and 1% from the current portfolio and within a 0.5% indifference level of the optimal efficient frontier portfolio. We examined subset (over 1000 portfolios) of random portfolios that were generated in the stated region around our current portfolio. The random portfolio that most closely met our criteria was then selected. The results are shown in Table 4:

Table 4

Asset	Current Portfolio	Optimal Efficient Frontier Portfolio	Recommended Portfolio
Domestic Inflation Linked Bonds	5.00%	15.00%%	6.83%
Domestic Bonds	15.00%	3.03%	11.21%
Cash	10.00%	0.00%	7.89%
Domestic Listed Equity	30.00%	32.22%	36.83%
Listed Property	10.00%	20.00%	14.55%
Foreign Equity	20.00%	0.00%	5.98%
Foreign Bonds	5.00%	18.93%	7.73%
Foreign Property	3.00%	8.25%	2.98%
Foreign Infrastructure	2.00%	10.00%	6.36%
Mean	12.37%	13.20%	12.95%
Stdev.	7.11%	7.11%	7.42%
Cost in bps	N/A	31 bps	18 bps
Statistical Indifference	N/A	0.68%	0.50%
Turnover	N/A	58% (29%)	40% (20%)
Net Improvement to returns	N/A	52 bps	40 bps
Probability of improvement to current portfolio	N/A	90.91%	93.23%
Probability of underperformance to current portfolio	N/A	10.09%	6.77%
Probability of exceeding net improvement in returns over current portfolio[2]	N/A	69.10%	67.68%

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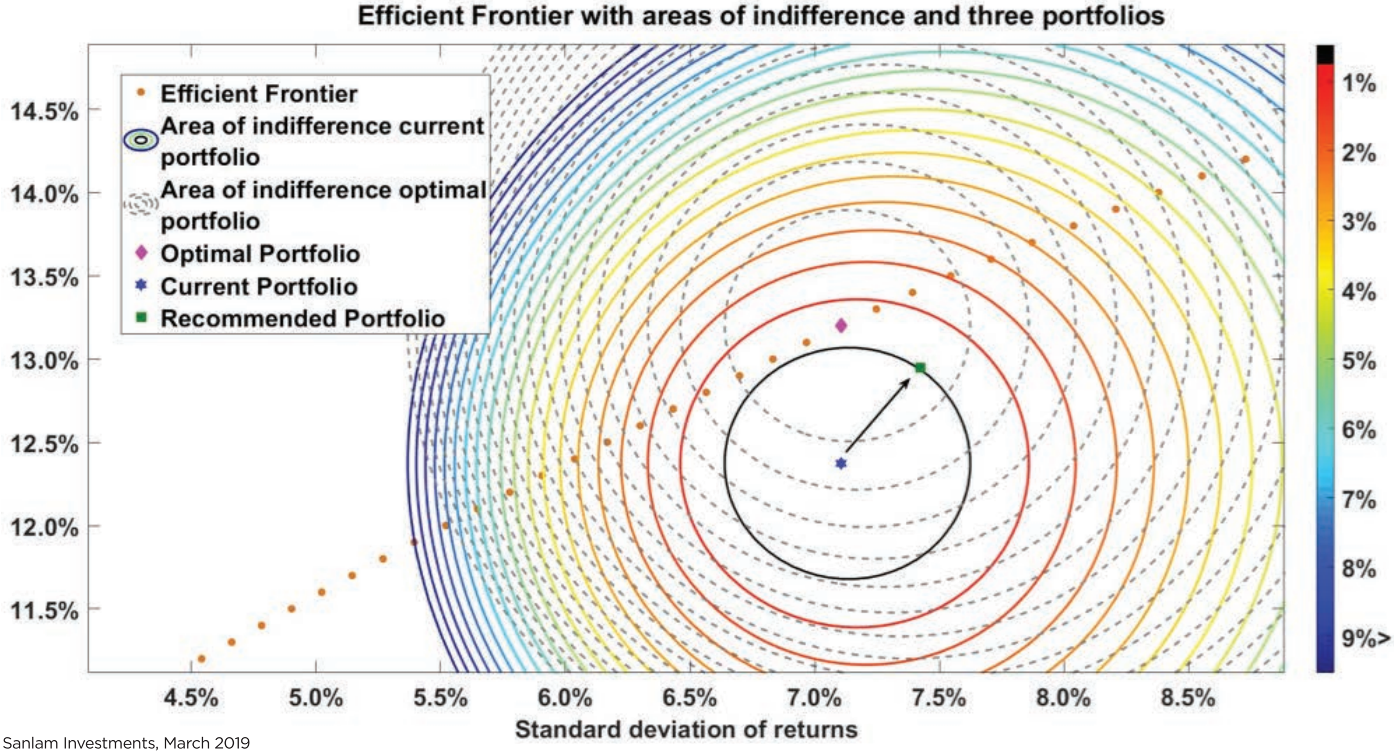
[2] By this probability we mean that given that in all possible situations, if our current portfolio gives a performance of x%, then what is the probability that the recommended portfolio will give at least x%+40bps (or 52bps for the efficient frontier portfolio). We calculate this probability by performing 100'000 Monte Carlo simulations on the portfolio returns over a period of 15 years and counting these incidences at the end of 15 years where, for the same random number seeds, the annualized performance in arithmetic terms of the improved portfolio is 75 bps or greater than the performance of the current portfolio.

From these results we can see that, without resorting to direct optimisation, random portfolios and areas of indifference can be used to create a portfolio that meets complex investment criteria and that shows an improvement over the current portfolio. Additionally, we also see that an improvement to the current portfolio does not always mean that the new portfolio should have a strictly lower standard deviation of return, as illustrated in Figure 1.3.

Figure 1.3



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Conclusion:

An area of indifference can be used to statistically show how different portfolios are. This measure is very useful in providing additional insights into portfolio construction. Although random portfolios help distinguish where the area of indifference is, an investor can always use specific weights and compare those against the current portfolio and the area of indifference in order to see how they may need to improve their current portfolio holdings. This methodology can then be combined with realistic cost and turnover constraints and a customised solution can then be derived to meet the unique objectives of an investor.

This methodology ensures the investor is made aware of all available opportunities, while still maintaining a relevant reference point (the efficient frontier portfolios) to base practical and realistic decisions on.

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Appendix:

In order to measure indifference in a quantitative way we utilise statistical divergence. Any measure of divergence, which we will refer to as f-divergence, takes a function D_f between different distributions, P and Q, that has to satisfy the following properties:

- Non-negativity (i.e. must always be positive and is equal to zero if and only if P and Q coincide.
- Monotonicity, i.e. any (transition probability) transformation κ that changes P and Q into P_κ and Q_κ should obey: $D_f(P \,||\, Q) \geq D_f(P_\kappa \,||\, Q_\kappa)$
- Joint convexity: i.e. for any $0 \leq \lambda \leq 1$ we have that
  - $D_f(\lambda P_1 + (1-\lambda)P_2 \,||\, \lambda Q_1 + (1-\lambda)Q_2) \leq \lambda D_f(P_1 \,||\, Q_1) + (1-\lambda) D_f(P_2 \,||\, Q_2)$

We can now formally define the divergence (or measure of indifference) between two probability distributions over a space Ω as:

$$D_f(P \,||Q) = \int_\Omega f\left(\frac{dP}{dQ}\right) dQ \tag{1}$$

This measure is only defined if the function f satisfies $f(1) = 0$. Depending on the function f, we can have various measures of indifference. The following (non-exhaustive list of) measures that can be used:

Divergence/Measure of indifference	**Function $f(t)$**
Kullback-Leibler	$t\log(t)$
Reverse Kullback-Leibler	$-\log(t)$
Hellinger distance	$\left(\sqrt{t}-1\right)^2, 2\left(1-\sqrt{t}\right)$
Pearson-Vajda χ_P^2	$(t-1)^2, t^2-1$
Jensen-Shannon	$-(t+1)\log\left(\frac{1+t}{2}\right) + t\log(t)$
α-divergence	$\begin{cases} \frac{4}{1-\alpha^2}\left(1-t^{\frac{1+\alpha}{2}}\right), & if\ \alpha \neq \pm 1 \\ t\ln(t), & if\ \alpha = 1 \\ -\ln(t), & if\ \alpha = -1 \end{cases}$

For example, if we use the Kullback-Leibler divergence measure then (1) becomes:

$$D(P \,||\, Q) = \int_{-\infty}^{\infty} p(x) \log\left(\frac{p(x)}{q(x)}\right) dx$$

The gold standard in literature is the Kullback-Leibler divergence however we will use the Bhattacharyya divergence (α-divergence with $\alpha = 0.5$) in the calculations presented in the indifference paper.

Note that $p(x)$ is the density function that is induced by combining a set of weights, w_1, in order to form a portfolio and we will compare this to $q(x)$ a density function induced by combining a set of weights w_2 in order to obtain the densities of the resulting portfolio. We can even replace $p(x)$ by the weighted